FLOW EQUATIONS AND THE CHIRAL PHASE TRANSITION

B.-J. SCHAEFER, O. BOHR AND J. WAMBACH

Institut für Kernphysik, TU Darmstadt, Schlossgartenstr. 9
D-64289 Darmstadt
Germany

Flow equations for an $O(N)$–symmetric effective potential are discussed and solved for the finite temperature case. The model is investigated at the critical point and critical exponents for various N are calculated.

1 Introduction

The N-component scalar field theory with an $O(N)$-symmetry often serves as a prototype for symmetry restoration investigations at finite temperature. The manifestation of universality of critical phenomena enables the applicability of the $O(N)$-symmetric model to a wide class of very different physical systems around the critical temperature.

Wilczek and Pisarski have argued that the two flavor chiral transition is either of second order or of first order depending on the $U_A(1)$ anomaly. In case that the transition is of second order Wilczek and Rajagopal have shown that two flavor QCD belongs to the same universality class as the three dimensional four component Heisenberg magnet [1]. Thus for $N = 4$ this theory is used as an effective model for the chiral phase transition in two flavor QCD enabling deeper insights into the chiral phase transition in QCD provided that the chiral order parameter remains close to thermal equilibrium through the transition. On the lattice it is still not clear whether QCD for two flavors and the $O(4)$-model lie in the same universality class [2].

In a second order transition, the system is at an infrared fixed point of the renormalization group. This means that the physics is scale invariant and the flow equations show a scaling solution. Due to a diverging correlation length in the chiral limit the order parameter fluctuates on all length scales and the correlation functions exhibit a power law behavior with critical exponents.

The article is organized as follows: In Sect. 2 we summarize the concept of the renormalization group (RG) method. In Sect. 3 we present some results for the $O(4)$-model at finite temperature. Finally Sect. 4 is dedicated to the critical behavior of the system at exactly the critical temperature.

2 Concept of the Renormalization Group Method

RG flow equations predict the behavior of a given theory in different momentum regimes. The main ingredient of the RG approach is the integration of irrelevant short-distance modes in order to derive a low-energy effective theory in the infrared in analogy to a discrete block-spin transformation on the lattice. For this purpose one introduces an infrared scale k that separates the fast-fluctuating short-distance modes from the slowly-varying modes. This idea – pioneered by Wilson and Kadanoff in the early seventies[3] – can be systematically incorporated e.g. in a momentum regularization in a transparent way. It results in an effective action parameterized by the averaged blocked field at the scale k. Thus the full one-particle irreducible (1PI) Feynman graphs or vertex functions are generated by the renormalized effective action in the limit $k \to 0$ and the k-dependent effective action provides a smooth interpolation between the bare action defined at the UV scale Λ, where no fluctuations are considered, and the renormalized effective action in the IR. The flow pattern of the given theory is obtained by studying an infinitesimal change of the IR scale k in the effective k-dependent action and is governed by differential flow equations.

Of course, the integration step cannot be performed in an exact way and one has to resort to some approximation such as the loop expansion. However a sharp momentum cutoff is in conflict with important underlying symmetries of the considered theory. The task is then to implement both the UV and the IR cutoff scales in a symmetry-conserving way, which can be accomplished by an operator cutoff regularization in Schwinger's proper-time representation.

The one-loop contribution to the effective action in general yields a non-local UV diverging logarithm, which can be rewritten in a proper-time representation and leads to

$$\Gamma = -\frac{1}{2} \int d^4x \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr} e^{-\tau(q^2 + V'')} \tag{1}$$

where the primes on the potential V denote differentiation w.r.t. ϕ. Here, the UV (IR) divergences appear for $\tau \to 0$ (∞). We modify the above expression by introducing a regulating *a priori* unknown smearing function $f_k(\tau)$ in the proper-time integrand. Differentiating the resulting k-dependent effective action w.r.t. the scale k yields the flow equations. It can be shown that all universal results in the infrared are independent of the choice of this regulating smearing function[4].

3   Finite Temperature

It is straightforward to generalize the above approach to finite temperature. Within the imaginary time (Matsubara) approach the four-dimensional theory at $T = 0$ is conveniently matched to the effectively three-dimensional behavior at the critical temperature by a replacement of the four-dimensional momentum integration in Eq. (1) with a three-dimensional integration and a Matsubara summation over the corresponding bosonic frequencies. Thus at finite temperature the scale k serves as a generalized IR cutoff for a combination of the three-dimensional momenta and Matsubara frequencies.

To solve the flow equations numerically we start the evolution in the broken phase deep in the UV region $k = \Lambda$ and use a tree-level parameterization of the potential with two initial values. These initial values are fixed at $T = 0$ and are constant in a relatively large temperature region.
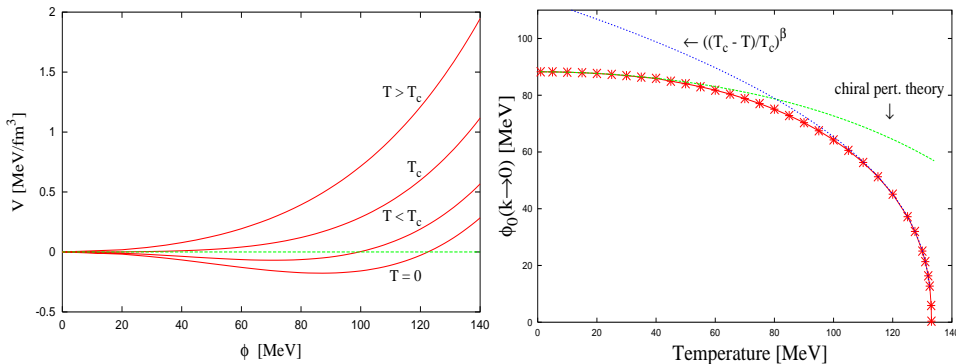

Figure 1:   Left panel: The potential for various temperatures. Right panel: The temperature-dependent order parameter in the chiral limit.

In the $k \rightarrow 0$ limit the temperature dependence of the potential $V(\phi)$ (left panel of Fig. 1) and the order parameter $\phi_{k\rightarrow 0}(T)$ (right panel) signal a second-order phase transition. Our calculations coincide with chiral perturbation theory up to $T \sim 45$ MeV. Around T_c we obtain a scaling behavior of the order parameter with a critical exponent $\beta \propto 0.4$ (cf. right panel).

4   Critical Behavior

At the critical temperature only properly rescaled quantities asymptotically exhibit scaling and the evolution of the system is purely three-dimensional [5]. This is the well-known dimensional reduction phenomenon. The rescaled dimensionless minimum $\kappa(t)$ of the potential approaches a constant (the fixed

point κ^*) as the dimensionless flow "time" $t = \ln(k/\Lambda)$ tends to $-\infty$ in the infrared (see left panel of Fig. 2). For a starting value near the critical value κ_{cr} at the ultraviolet scale $t = 0$ the evolution moves towards either the spontaneously broken $\kappa \neq 0$ or the symmetric $\kappa = 0$ phase in the infrared. Due to the rescaling, which in the infrared tends to a constant value or zero during the k-evolution, the dimensionless minimum $\kappa = \rho_0/k^2$ diverges for $k \to 0$ in the broken phase. The time the system spends on this scaling solution can be rendered arbitrarily long by appropriate fine tuning of the initial values at $t = 0$.
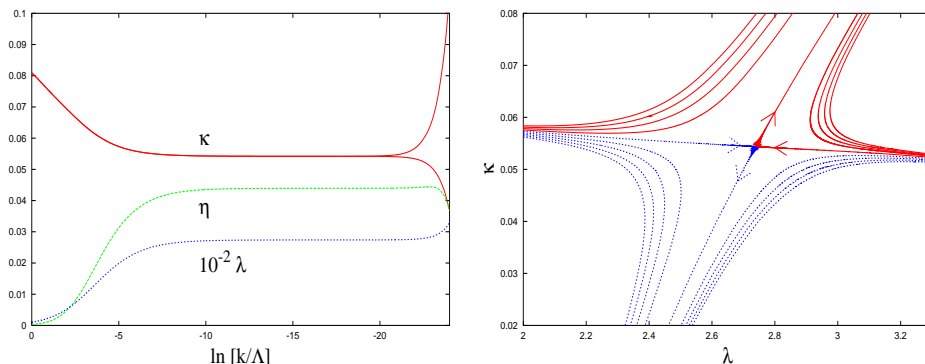

Figure 2: Left panel: The minimum κ, η and coupling λ as function of t. Right panel: The phase diagram for $N = 1$ (see text for details).

Besides of the trivial high-temperature and low-temperature fixed points the $O(N)$-symmetric model exhibits a nontrivial mixed fixed point inherent in the flow equations. The region around this fixed point is displayed in right panel of Fig. 2 where the rescaled dimensionless quantities κ and λ are plotted for $N = 1$ and for different initial values during the evolution toward the infrared.

In general there are two relevant physical parameters for the $O(N)$-model which must be adjusted to bring the system to the critical fixed point according to universality class arguments. Since we here work in the chiral limit (no external sources or masses) only one relevant eigenvalue from the linearized renormalization group equations is left (the temperature). This can also be seen in Fig. 2. The quantity κ is the relevant variable because repeated renormalization group iterations (which are the discrete analog to the continuous evolution with respect to $t \to -\infty$) drive the variable away from the fixed point value while λ is the irrelevant variable being iterated toward the fixed point if the initial values are chosen sufficiently close to it. Thus there is a one-

4

dimensional curve of points attracted to the fixed point which is the so-called critical surface for this model.

One nicely sees that this critical line separates both phases. When we choose initial values near the critical line the system spends a long time near the critical point. E.g. starting with $\kappa > \kappa_{cr}$ and λ finite the evolution ultimately tends to the zero-temperature fixed point corresponding to very large values of κ. On the other hand starting below the critical line the evolution of κ tends towards the high-temperature fixed point corresponding to small κ's. Only for exactly $\kappa = \kappa_{cr}$ at $t = 0$ the renormalization group trajectory flows into the critical mixed fixed point, which means that the long distance behavior at the critical point is the same as that of the fixed point.

Here the predictive power of this nonperturbative approach becomes visible: During the evolution near the scaling solution the system loses memory of the initial starting value in the UV and the effective three-dimensional dynamics near the transition is completely determined by the fixed point being independent of the details of the considered microscopic interaction at short distances. Exactly at a second-order phase transition the potential should be described by a k-independent solution. The equation of state drives the potential away from the critical temperature. As a result, after the potential has evolved away from the scaling solution, its shape is independent of the choice of the initial values and exhibits universal behavior.

In the vicinity of the critical point we obtain a scaling behavior of the system governed by critical exponents which parameterize the singular behavior of the free energy around the phase transition. Out of the six critical exponents for the $O(N)$-model only two are independent due to four scaling relations which we have verified by explicit calculations of α, β, γ and ν in three dimensions for various N (see tab. 1).

The critical exponent β parameterizes the behavior of the spontaneous magnetization or the order parameter for the broken phase ($\kappa_\Lambda > \kappa_{cr}$) in the vicinity of T_c. The difference $\kappa_\Lambda - \kappa_{cr}$ is a measure of the distance from the phase transition irrespective of any given value of λ_Λ. If κ_Λ is interpreted as a function of temperature this difference is $\propto (T - T_c)$, i.e. κ_{cr} defines the critical temperature in three dimensions.

All calculated critical exponents nicely converge to the large–N values. In addition for $N = 1$ we have found $\eta = 0.0439$ and $\delta = 4.748$.

The structure of the regularized RG-improved flow equations depends on the used *a priori* unknown cutoff functions f_k. Employing the same class of the cutoff function as in ref.[4] we have calculated critical exponents for several cutoff functions with more and more monomials on a grid for the full potential. A small systematic decrease in the values for the critical exponents with the num-

N	ν	β	α	γ
N=1	0.643	0.335	0.071	1.258
N=2	0.695	0.3475	-0.085	1.39
N=3	0.75	0.375	-0.25	1.50
N=4	0.79	0.395	-0.37	1.58
N=10	0.911	0.4555	-0.733	1.822
N=100	0.993	0.4965	-0.979	1.986
large–N	1.0	0.5	-1.0	2.0

Table 1: Critical exponents for different N compared with the large–N result.

ber of included monomials of the smearing function is observed. Restricting to less monomials in the cutoff function f_k seems to accelerate the IR convergence at the critical point. This is visible in the evolution of the minimum of the potential which stays longer on the critical trajectory in the vicinity of the scaling solution. For more monomials the evolution of the minimum escapes faster from the critical trajectory to the symmetric as well as the broken phase.

To conclude we have presented a powerful nonperturbative method based on the RG approach applied to the $O(N)$-model at finite temperature, which, in fact, is consistent with pertinent lattice simulations. It remains to be seen in how far the critical $O(4)$-behavior is realized in full two flavor QCD simulations.

Acknowledgments

References

1. R. Pisarski and F. Wilczek, *Phys. Rev.* D **29**, 338 (1984); K. Rajagopal and F. Wilczek, *Nucl. Phys.* B **399**, 395 (1993).
2. E. Laermann, these proceedings.
3. L. P. Kadanoff, *Physica* **2** (1966) 263; K. G. Wilson, *Phys. Rev.* B **4**, 3174,3184 (1971).
4. G. Papp, B.-J. Schaefer, H.J. Pirner and J. Wambach, `hep-ph/9909246`.
5. J. Berges, D.-U. Jungnickel and C. Wetterich, *Phys. Rev.* D **59**, 034010 (1999).